Math 217 Fall 2025

Quiz 22 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) Let V be a d-dimensional vector space, with two bases $\mathcal{B} = (b_1, \ldots, b_d)$ and $\mathcal{A} = (a_1, \ldots, a_d)$. The *change-of-basis matrix from $\mathcal{B}$ to $\mathcal{A}$* is ...

   Solution: The unique $d \times d$ matrix $S_{\mathcal{B}\to\mathcal{A}}$ that converts $\mathcal{B}$-coordinates into $\mathcal{A}$-coordinates; i.e., for every $v \in V$,
   $$[v]_{\mathcal{A}} = S_{\mathcal{B}\to\mathcal{A}}\,[v]_{\mathcal{B}}.$$
   Equivalently, the j-th column of $S_{\mathcal{B}\to\mathcal{A}}$ is the $\mathcal{A}$-coordinate vector of b_j:
   $$S_{\mathcal{B}\to\mathcal{A}} = \begin{bmatrix} | & & | \\ [b_1]_{\mathcal{A}} & \cdots & [b_d]_{\mathcal{A}} \\ | & & | \end{bmatrix}.$$

   (b) Let $T : V \to V$ be a linear transformation of a vector space V and let $\mathfrak{B} = (v_1, \ldots, v_n)$ be an ordered basis of V. The *matrix of T with respect to $\mathfrak{B}$* is ...

   Solution: The $n \times n$ matrix $[T]_{\mathfrak{B}}$ whose j-th column is the $\mathfrak{B}$-coordinate vector of $T(v_j)$; i.e.,
   $$[T]_{\mathfrak{B}} = \begin{bmatrix} | & & | \\ [T(v_1)]_{\mathfrak{B}} & \cdots & [T(v_n)]_{\mathfrak{B}} \\ | & & | \end{bmatrix}.$$
   Equivalently, for every $v \in V$, $[T(v)]_{\mathfrak{B}} = [T]_{\mathfrak{B}}\,[v]_{\mathfrak{B}}$.

2. Suppose V is a vector space and $v_0 \in V$. Suppose also that $v_1, \ldots, v_m$ are linearly independent vectors in V. Show that
$$v_0 \in \operatorname{Span}(v_1, \ldots, v_m) \iff \{v_0, v_1, \ldots, v_m\} \text{ is linearly dependent.}$$

Solution: ($\Rightarrow$) If $v_0 = \sum_{i=1}^m c_i v_i$ for some scalars c_i, then
$$v_0 - \sum_{i=1}^m c_i v_i = 0$$

*For full credit, please write out fully what you mean instead of using shorthand phrases.

is a nontrivial linear relation among $v_0, v_1, \ldots, v_m$, so they are linearly dependent.

($\Leftarrow$) If $v_0, v_1, \ldots, v_m$ are linearly dependent, there exist scalars $a_0, a_1, \ldots, a_m$, not all zero, with $a_0 v_0 + \sum_{i=1}^m a_i v_i = 0$. If $a_0 = 0$, then $\sum_{i=1}^m a_i v_i = 0$ is a nontrivial relation among $v_1, \ldots, v_m$, contradicting their independence. Hence $a_0 \neq 0$, and

$$v_0 = -\sum_{i=1}^{m} \frac{a_i}{a_0} v_i \in \operatorname{Span}(v_1, \ldots, v_m).$$

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

(a) Let $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ be an orthonormal ordered basis for $\mathbb{R}^n$. If $\vec{x} \in \mathbb{R}^n$, then

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} \vec{x} \cdot \vec{v}_1 \\ \vec{x} \cdot \vec{v}_2 \\ \vdots \\ \vec{x} \cdot \vec{v}_n \end{bmatrix}.$$

Solution: TRUE. Writing $\vec{x} = \sum_{j=1}^n c_j \vec{v}_j$ and taking dot products with $\vec{v}_i$, $\vec{x} \cdot \vec{v}_i = \sum_{j=1}^n c_j (\vec{v}_j \cdot \vec{v}_i) = c_i$ since $\mathcal{B}$ is orthonormal. Thus the $\mathcal{B}$-coordinates are the listed dot products.

(b) For any $S \subseteq \mathbb{R}^n$, $S^\perp$ is a subspace of $\mathbb{R}^n$.

Solution: TRUE. By definition, $S^\perp = \{x \in \mathbb{R}^n : \ x \cdot s = 0 \text{ for all } s \in S\}$. We have $0 \in S^\perp$ since $x \cdot 0 = 0$, and if $x, y \in S^\perp$ and $\alpha, \beta \in \mathbb{R}$, then for all $s \in S$, $(\alpha x + \beta y) \cdot s = \alpha(x \cdot s) + \beta(y \cdot s) = 0$, so $\alpha x + \beta y \in S^\perp$. Hence $S^\perp$ is a subspace.